KASTELEYN'S THEOREM ON DOMINO TILINGS

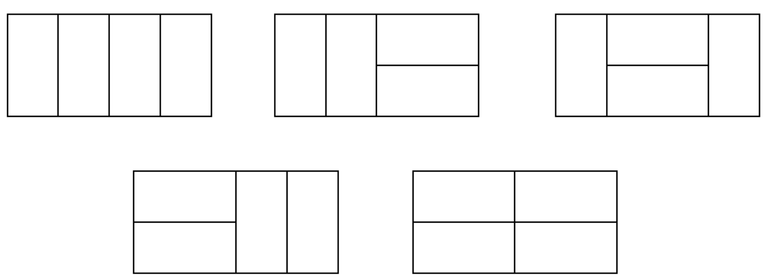
AARON KAUFER

Abstract. This paper answers the question of "How many ways are there to tile an $n \times m$ rectangle with 2×1 dominos?" Specifically, it builds up all the tools necessary to prove Kasteleyn's Theorem, which establishes that the number of tilings is $\sqrt{|\det(K)|}$, where K is the adjacency matrix of the $n \times m$ grid graph with horizonal weight 1 and vertical weight $i = \sqrt{-1}$. From there, $\det(K)$ is calculated by determining its eigenvalues and multipling them together, leading to the final explicit solution:

$$T_{n,m} = \prod_{j=1}^{n} \prod_{k=1}^{m} \left(4\cos\left(\frac{\pi j}{n+1}\right)^2 + 4\cos\left(\frac{\pi k}{m+1}\right)^2 \right)^{1/4}$$

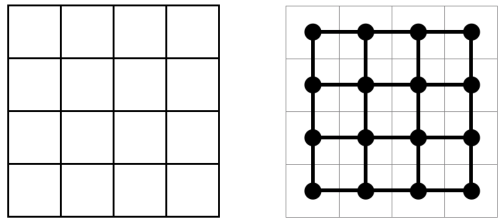
1. Introduction

In this paper, we study the general problem of *Domino Tilings*. That is, given an $n \times m$ grid, how many different ways can we tile it with 2×1 rectangles? For example, there are five different ways to tile a 2×4 rectangle, as shown below:



For the remainder of this paper, we denote the number of domino tilings of an $n \times m$ rectangle by $T_{n,m}$. As shown above, $T_{2,4} = 5$.

To begin this problem, we translate it into the language of graph theory. Given an $n \times m$ grid, we define the $n \times m$ *grid graph* (denoted $G_{n,m}$) as the graph whose vertices are squares on the $n \times m$ grid, and two vertices are connected if the corresponding squares are adjacent. The vertices of $G_{n,m}$ can be thought of as points in $\mathbb{Z}^2$, where (for the sake of convinience) the bottom-left vertex is given the coordinates $(1, 1)$. For example, the following image depicts a 4×4 grid and its corresponding grid graph:



From this, we can see that a 2×1 domino corresponds to an edge on the graph $G_{n,m}$, and a domino tiling corresponds to a collection of non-overlapping edges of $G_{n,m}$ which cover the entire graph. This gives rise to the following definition:

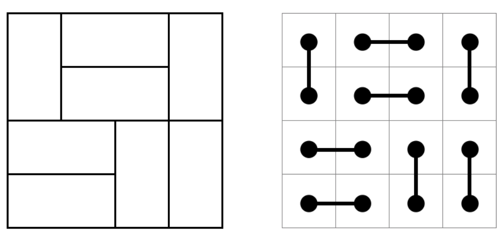
Definition 1.1. Let G be a graph. Then a *perfect matching* M of G is a collection of edges such that no two edges in M share a vertex, and every vertex in G touches some edge in M.

From this definition, we get the following theorem which serves as the starting point for our analysis:

Theorem 1.2. *The number of domino tilings of an $n \times m$ rectangle is equal to the number of perfect matchings of the $n \times m$ grid graph $G_{n,m}$.*

Proof. This follows straightforwardly from the definition. □

To see the correspondence between perfect matchings and domino tilings, we give an example:

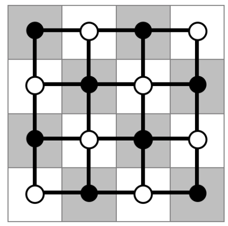


One observation about the graph $G_{n,m}$ is that it can be "colored" black and white like a chessboard. To be more concrete, we have the following definition:

Definition 1.3. A graph G is said to be *bipartite* if it canbe divided into two "parts". That is, if there exists a way to partition the set of verticies of G into V_1 and V_2 (both nonempty) such that every edge in G flows from a vertex in V_1 to a vertex in V_2.

Theorem 1.4. *The grid graph $G_{n,m}$ is bipartite.*

Proof. This proof is best done via an illustration:



In this picture, the black verticies correspond to the set V_1 and the white verticies correspond to the set V_2. □

2. Kasteleyn's Theorem

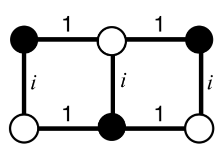
In this section, we state and prove a theorem first proven by Pieter Kasteleyn in 1961 which gives a rather elegant way of counting the number of perfect matchings of $G_{n,m}$. We first give a few definitions:

Definition 2.1. If u and v are two verticies in $G_{n,m}$, then we define the *Kasteleyn weight* $w(u,v)$ as:

$$w(u,v) = \begin{cases} 1 & (u,v) \text{ is a horizontal edge} \\ i & (u,v) \text{ is a vertical edge} \\ 0 & \text{else} \end{cases}$$

Definition 2.2. We define the Kasteleyn-weighted grid graph $G'_{n,m}$ to be the grid graph $G_{n,m}$, where the edges are weighted according to the Kasteleyn weight.

For example, the following illustrates the graph $G'_{2,3}$:



We recall the notion of an adjacency matrix of a weighted graph:

Definition 2.3. If G is a weighted graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and weight function $w : V \times V \to \mathbb{C}$, then the $n \times n$ adjacency matrix A is defined by $A_{i,j} = w(v_i, v_j)$. Furthermore, if $x, y \in V$, such that $x = v_i$ and $y = v_j$, then $A_{x,y}$ is taken to be equal to $A_{i,j}$ (in other words, $A_{u,v} = w(u,v)$).

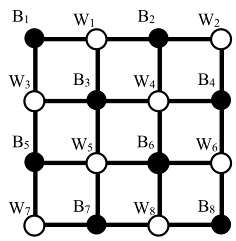
One small detail about this definition is that it depends on the ordering $v_1, v_2, \ldots, v_n$ of the verticies, so different orderings give rise to different matrices. However, rearranging the order of the vertices simply has the effect of conjugating the matrix by a permutation matrix, so the properties of the matrix that we care about, specifically the eigenvalues and determinant, are not dependent on the ordering. Hence, expressions like "the eigenvalues of a graph" or "the determinant of a graph" are unambiguous.

We are now in a position to state Kasteleyn's Theorem:

Theorem 2.4. *Let $K_{n,m}$ be the adjacency matrix of the Kasteleyn-weighted grid graph $G'_{n,m}$. Then the number of perfect matchings of the grid graph $G_{n,m}$ is equal to $\sqrt{|\det(K_{n,m})|}$.*

We will prove this result through a series of intermediate steps. From here on, m and n are assumed to be fixed, so we define $K = K_{n,m}$, $G' = G'_{n,m}$, and $G = G_{n,m}$.

First, we investigate the general structure of the matrix K. First, we recall that G is a bipartite graph, so we color the vertices of the graph as black or white according to "chessboard" coloring, and we label the vertices as $B_1, B_2, \dots, B_N$ and $W_1, W_2, \dots, W_N$, where $N = nm/2$ (The number of dominos that can fit on the $n \times m$ grid). For example, when $n = m = 4$, the labeling looks like:



If we order the vertices of G as $B_1, \dots, B_N, W_1, \dots, W_N$ and contruct the adjacency marix K, we can see that because the matrix must take the form:

$$K = \begin{bmatrix} \mathbf{0} & A \\ A^T & \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ is the $N \times N$ matrix of all zeros, and A is an $N \times N$ matrix. The reason for this is becuase there is no edge between a black vertex and a white vertex, and in addition, for all vertices u, v, we have $w(u,v) = w(v,u)$, so K must be symmetric.

By using standard properties of determinants, we get:

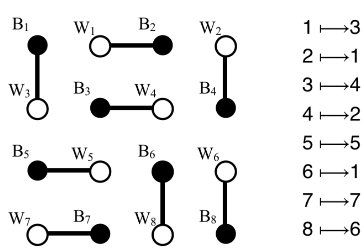
$$|\det(K)| = \left|\det\left(-AA^T\right)\right| = |\det(A)|^2$$

Thus, $|\det(A)| = \sqrt{|\det(K)|}$ and so in order to prove Kasteleyn's Theorem, it suffices to prove that $|\det(A)|$ is equal to the number of perfect matchings of G.

It is useful now to examine the structure of A. Directly from the definition, we can see that:

$$A_{i,j} = \begin{cases} 1 & (B_i, W_j) \text{ is a horizontal edge in } G \\ i & (B_i, W_j) \text{ is a vertical edge in } G \\ 0 & \text{else} \end{cases}$$

We now consider an arbitrary perfect matching M of the graph G. Since G is bipartite, each edge in M must connect some black vertex B_i to a white vertex W_j. From this, we can construct a permutation $\sigma_M \in S_N$ defined by $\sigma_M(i) = j$ if (B_i, W_j) is an edge in M. The following illustrates a perfect matching and its corresponding permutation:



If a permutation σ gives rise to a perfect matching M as shown above, then we say that σ *represents* M. Although every perfect matching of G is represented by a permutation in S_N, the converse is not true. For example, in the diagram above, it would clearly be impossible for a permutation σ_M that satisfies $\sigma_M(1) = 8$ to represent a perfect matching of G, because it would imply that (B_1, W_8) is an edge in G, which is false.

The question of determining when a permutation represents a perfect matching is answered by the following theorem:

Theorem 2.5. *If $\sigma \in S_N$, then the set of edges of G defined by $\{(B_i, W_{\sigma(i)}) \mid i = 1 \dots N)\}$ is a perfect matching on G if and only if:*

$$A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{N,\sigma(N)} \neq 0$$

Proof. ($\Longrightarrow$) If $A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{N,\sigma(N)} = 0$, then $A_{i,\sigma(i)} = 0$ for at least one i, which means that $(B_i, W_{\sigma(i)})$ is not an edge of G, so the given set of edges cannot be a perfect matching.

($\Longleftarrow$) If the given product is nonzero, then every term in the product is also nonzero, which means that for each i, the edge $(B_i, W_{\sigma(i)})$ is an edge in G. Since G is bipartite, these edges cover the entire graph, meaning that it is a perfect matching. □

In addition, we notice that since $A_{i,j} \in \{0, 1, i\}$, it follows that:

$$\left|A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{N,\sigma(N)}\right| = \begin{cases} 1 & \sigma \text{ represents a perfect matching in } G \\ 0 & \sigma \text{ does not represent a perfect matching in } G \end{cases}$$

Hence, because $T_{n,m}$ is the number of perfect matchings of G, we can see:

$$\sum_{\sigma \in S_N} \left|A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{N,\sigma(N)}\right| = T_{n,m} \tag{2.1}$$

Some readers may recognize the left hand side of (2.1) as being similar to the permutation definition of the determinant:

$$\det(A) = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{N,\sigma(N)} \tag{2.2}$$

Since we have shown that proving Kasteleyn's Theorem is equivalent to proving $|\det(A)| = T_{n,m}$, we want to show that none of the terms in (2.2) "cancel" each other out. In order to do this, we make the following definition:

Definition 2.6. Let M be a perfect matching of G, and let $\sigma \in S_N$ be the permutation that represents M. Then we define the *sign* of M as:

$$\operatorname{sgn}(M) := \operatorname{sgn}(\sigma) A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{N,\sigma(N)}$$

Since $\operatorname{sgn}(M)$ can only take on the values $1, -1, i$, and $-i$, it follows that $|\operatorname{sgn}(M)| = 1$ for all M. Hence, if we can prove that for any two arbitrary perfect matchings M and M' of G have the same sign (i.e. $\operatorname{sgn}(M) = \operatorname{sgn}(M')$), then we will have proven Kasteleyn's Theorem. This can be seen by letting S be the set of all perfect matchings of G and rewriting (2.2) as:

$$|\det(A)| = \left|\sum_{M \in S} \operatorname{sgn}(M)\right| = |T_{n,m} \cdot \operatorname{sgn}(M_0)| = T_{n,m}$$

where M_0 is any arbitrary perfect matching. Hence, it suffices to prove:

Theorem 2.7. *If M and M' are two abritrary perfect matchings of G, then*

$$\operatorname{sgn}(M) = \operatorname{sgn}(M')$$

The proof of this theorem boils down to a tedious proof by induction on certain properties of the matchings M and M'. Because it is not very enlightening, the reader is referred to [2] for a proof of this theorem.

3. Eigenvalues of the Grid Graph

Now that we have established $T_{n,m} = \sqrt{|\det(K_{n,m})|}$, it remains to calculate $\det(K_{n,m})$, which we will do by determining its eigenvalues and multiplying them together. Since, as we established earlier, "the eigenvalues of a graph" is unambiguous, we shall speak of the eigenvalues of $G'_{n,m}$ instead of $K_{n,m}$. Our work will be composed of roughly three steps:

(1) Define the cartesian product of two graphs and identify G' as the cartesian product of two line graphs.
(2) Determine the eigenvalues of a line graph.
(3) Determine how the eigenvalues of two graphs relate to the eigenvalues of their cartesian product.

We begin with the first part.

Definition 3.1. If A and B are two graphs with vertex sets U and V respectively, then the *cartesian product* of A and B (denoted $A \square B$) is the graph with vertex set $U \times V$ with edges defined by:

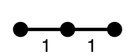
$$(u_1, v_1) \sim (u_2, v_2) \quad \text{iff} \quad \begin{cases} v_1 = v_2 \quad \text{and} \quad u_1 \sim u_2 \text{ (in the graph A)} \\ \quad \text{or} \\ u_1 = u_2 \quad \text{and} \quad v_1 \sim v_2 \text{ (in the graph B)} \end{cases}$$

where $\sim$ denotes adjacency.

Furthermore, if A and B are weighted graphs with weight functions w_A and w_B respectively, then $A \square B$ is a weighted graph with weight function:

$$w_{A \square B}((u_1, v_1), (u_2, v_2)) = \begin{cases} w_B(v_1, v_2) & u_1 = u_2 \\ w_A(u_1, u_2) & v_1 = v_2 \\ 0 & \text{else} \end{cases}$$

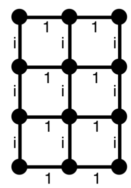
To illustrate this definition and how it lines up with $G'_{n,m}$, we define H_m as the horizonaltal line graph of length n with edge weights 1. Explicitly, the vertices of H_m are the numbers $\{1, 2, \ldots, m\}$, and two numbers are adjacent if they are consective numbers. For example, the graph of H_3 looks like:



We now define V_n as the vertical line graph of length m with edge weights i. For example, V_4 looks like:

The reader should note that we drew V_4 vertically for asthetic purposes, and it could have alternatively been drawn horizontally.

We now draw the graph $V_4 \square H_3$:



The reader should verify that the edges and edge weights line up with the definition of cartesian product of two graphs. Furthermore, the reader should recognize that this graph is identical to the graph $G'_{4,3}$. To this extent, the following theorem should be obvious:

Theorem 3.2. *The graphs $G'_{n,m}$ and $V_n \square H_m$ are isomorphic as weighted graphs and hence have the same eigenvalues and determinant.*

We now wish to find the eigenvalues of H_m (the methods will also work for V_m). In particular, if we examine the graph H_m, then we can quite easily what its adjacency matrix looks like. In particuar, it is the $m \times m$ matrix of the form:

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is convininent to often talk in terms of functions instead of vectors, so we will say a vector $\mathbf{v} \in \mathbb{C}^m$ is represented by $f : \{1, 2, \ldots, m\} \to \mathbb{C}$ when $\mathbf{v}$ is of the form $\mathbf{v} = (f(1), f(2), \ldots, f(m))^T$ so that $\mathbf{v}_x = f(x)$ for $x \in \{1, 2, \ldots, m\}$. By examining the matrix B, we can see that for $1 < x < m$, its action on an arbitrary vector $\mathbf{v}$ represented by f is:

$$(B\mathbf{v})_x = f(x-1) + f(x+1) \tag{3.1}$$

Furthermore, $(B\mathbf{v})_1 = f(2)$ and $(B\mathbf{v})_m = f(m-1)$.

We now fix $j \in \{1, 2, \ldots, m\}$ and define $f_j(x) = \sin\left(\frac{\pi j x}{m+1}\right)$. It is easily checked that $f_j(0) = 0$ and $f_j(n+1) = 0$, so for the vector $\mathbf{v}$ represented by f_j, the equation (3.1) holds for all $1 \le x \le m$.

Theorem 3.3. *For fixed $j \in \{1, 2, \ldots, m\}$, the vector $\mathbf{v}$ represented by f_j is an eigenvector of the matrix B with eigenvalue $\lambda_j = 2\cos\left(\frac{\pi j}{m+1}\right)$.*

Proof. The equation $B\mathbf{v} = \lambda_j \mathbf{v}$ is equivalent to the statement that for all $x \in \{1, 2, \dots, m\}$, $(B\mathbf{v})_x = \lambda_j \mathbf{v}_x$. To simplify calculations, we let $\alpha = \frac{\pi j}{m+1}$ so that $f_j(x) = \sin(\alpha x)$. Then, using standard trig identities, we can compute:

$$\begin{aligned}\frac{(B\mathbf{v})_x}{\mathbf{v}_x} &= \frac{f(x-1)+f(x+1)}{f(x)} \\ &= \frac{\sin(\alpha x - \alpha) + \sin(\alpha x + \alpha)}{\sin(\alpha x)} \\ &= \frac{2\sin(\alpha x)\cos(\alpha)}{\sin(\alpha x)} \\ &= 2\cos(\alpha) = \lambda_j\end{aligned}$$

□

Corollary 3.4. *The eigenvalues of the horizontal line graph H_m are the values $\lambda_j = 2\cos\left(\frac{\pi j}{m+1}\right)$ for $j \in \{1, 2, \dots, m\}$.*

Proof. That fact that each λ_j is an eigenvalue was established by the previous theorem. To see that these make up all eigenvalues, just note that each λ_j is distinct and that an $m \times m$ matrix cannot have more than n eigenvalues. □

Corollary 3.5. *The eigenvalues of the vertical line graph V_n (with edge weight i) are $\mu_k = 2i\cos\left(\frac{\pi k}{n+1}\right)$ for $k \in \{1, 2, \dots, n\}$.*

Proof. The proof is essentially identical to that of Theorem 3.3, except the matrix B is replaced with the matrix iB. □

We now wish to relate the eigenvalues of $G'_{n,m} = V_n \Box H_m$ to those of V_n and H_m. For this, we appeal to the more general theorem:

Theorem 3.6. *Let A and B be weighted graphs with vertex sets $\{1, 2, \dots, n\}$ and $\{1, 2, \dots, m\}$ respectively. If λ is an eigenvalue of A, and μ is an eigenvalue of B, then $\lambda + \mu$ is an eigenvalue of $A \Box B$.*

Proof. To ease notation, let $[A], [B]$, and $[A \Box B]$ be the adjacency matrices of A, B, and $A \Box B$ respectively, and define $[n] := \{1, 2, \dots, n\}$ (and likewise for $[m]$). Now, let $\mathbf{a}$ be the eigenvector of $[A]$ corresponding to the eigenvalue λ and let $\mathbf{b}$ be the eigenvector of $[B]$ corresponding to the eigenvalue μ. Furthmore, let $g : [n] \to \mathbb{C}$ and $h : [m] \to \mathbb{C}$ be the functions that represent the vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Since $\mathbf{a}$ and $\mathbf{b}$ are eigenvectors, we get that $[A]\mathbf{a} = \lambda \mathbf{a}$ and $[B]\mathbf{b} = \mu \mathbf{b}$, which is equivalent to saying that for all $x \in [n]$ and $y \in [m]$,

$$([A]\mathbf{a})_x = \lambda \mathbf{a}_x \text{ and } ([B]\mathbf{b})_y = \mu \mathbf{b}_y$$

By writing out the explicit expressions for matrix multiplication, we see:

$$([A]\mathbf{a})_x = \sum_{u \in [n]} [A]_{x,u} \mathbf{a}_u = \sum_{u \in [n]} [A]_{x,u} g(u)$$

$$([B]\mathbf{b})_y = \sum_{v \in [m]} [B]_{y,v} \mathbf{b}_v = \sum_{v \in [n]} [A]_{y,v} h(v)$$

We note that the matrix $[A \Box B]$ is an $nm \times nm$ matrix, so the vectors it acts on are elements of $\mathbb{C}^{nm}$. As before, we shall associate a vector $\mathbf{w} \in \mathbb{C}^{nm}$ via a function $f : [n] \times [m] \to \mathbb{C}$. Using the functions g and h defined earlier, we define the function $f : [n] \times [m] \to \mathbb{C}$ as $f(x, y) = g(x)h(y)$, and we let $\mathbf{w}$ be the vector represented by f (that is, $\mathbf{w}_{(x,y)} = f(x, y) = g(x)h(y)$). We claim that $\mathbf{w}$ is an eigenvector of $[A\Box B]$ with eigenvalue $\lambda + \mu$.

In order to verify this, we first explicity write out what the matrix $[A\Box B]$ looks like, which follows directly from its definition:

$$[A\Box B]_{(x,y),(u,v)} = \begin{cases} [A]_{x,u} & y = v \\ [B]_{y,v} & x = u \\ 0 & \text{else} \end{cases}$$

We now write out, through explicit matrix multiplication, the action of $[A\Box B]$ on the vector $\mathbf{w}$:

$$\begin{aligned} ([A\Box B]\mathbf{w})_{(x,y)} &= \sum_{(u,v)\in[n]\times[m]} [A\Box B]_{(x,y),(u,v)} \mathbf{w}_{(u,v)} \\ &= \sum_{(u,v)\in[n]\times[m]} [A\Box B]_{(x,y),(u,v)} g(u)h(v) \\ &= \sum_{\substack{u\in[n] \\ v=y}} [A\Box B]_{(x,y),(u,v)} g(u)h(v) + \sum_{\substack{v\in[m] \\ u=x}} [A\Box B]_{(x,y),(u,v)} g(u)h(v) \\ &= h(y) \sum_{u\in[n]} [A]_{x,u} g(u) + g(x) \sum_{v\in[m]} [B]_{y,v} h(v) \\ &= h(y)([A]\mathbf{a})_x + g(x)([B]\mathbf{b})_y \\ &= h(y)\lambda g(x) + g(x)\mu h(y) = (\lambda + \mu)g(x)h(y) = (\lambda + \mu)\mathbf{w}_{(x,y)} \end{aligned}$$

Thus, $\mathbf{w}$ is an eigenvector of $[A\Box B]$ with eigenvalue $\lambda + \mu$.

□

Corollary 3.7. *The eigenvalues of the graph $G'_{n,m}$ (and of the matrix $K_{n,m}$) are precisely the numbers:*

$$2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)$$

for $j \in \{1, 2, \dots, m\}$ and $k \in \{1, 2, \dots, n\}$.

Proof. The proof is immediate from the previous Theorem together with Theorems 3.3 and 3.5. □

Corollary 3.8.

$$T_{n,m} = \prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos\left(\frac{\pi j}{m+1}\right)^2 + 4\cos\left(\frac{\pi k}{n+1}\right)^2\right)^{1/4}$$

Proof. We know, by Kasteleyn's Theorem, that $T_{n,m} = \sqrt{|\det(K_{n,m})|}$. Since the determinant is equal to the product of the eigenvalues, we get:

$$\begin{aligned}
T_{n,m} &= |\det(K_{n,m})|^{1/2} \\
&= \left| \prod_{j=1}^{m} \prod_{k=1}^{n} \left(2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)\right) \right|^{1/2} \\
&= \prod_{j=1}^{m} \prod_{k=1}^{n} \left| 2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right) \right|^{1/2}
\end{aligned}$$

which is easily seen to be equivalent to the desired expression. □

References

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